APPENDIX 2

IX-MATHEMATICS-NCERT-2023-24

INTRODUCTION TO MATHEMATICAL MODELLING

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1. A mathematical model is a mathematical relation that describes some real-life situation.
2. Mathematical models are used to solve many real-life situations like:
   - launching a satellite.
   - predicting the arrival of the monsoon
   - controlling pollution due to vehicles
   - reducing traffic jams in big cities

The process of mathematical modelling , its Advantages and Limitations

Step 1 : Formulation: Formulation involves the following three steps

(i) Stating the problem.

(ii) Identifying relevant factors.

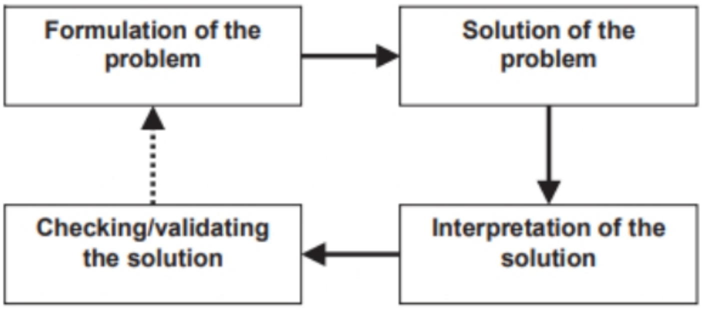
(iii) Mathematical Description,

Step 2 : Finding the solution.

Step 3 : Interpretating the solution.

Step 4 : Validating the solution.

A summary of the order in which the steps in mathematical modelling are carried out.



Example 1 : I travelled 432 kilometres on 48 litres of petrol in my car. I have to go by my car to a place which is 180 km away. How much petrol do I need?

Sol: **Formulation**: The amount of petrol we need varies directly with the distance we travel.

Petrol needed for travelling 432 km = 48 litres

Mathematical Description: Let x = distance I travel y = petrol I need

y varies directly with x

y = kx, where k is a constant.

y = 48, x = 432

$$k = \frac{y}{x} = \frac{48}{432} = \frac{1}{9}$$

$$y = kx \Rightarrow y = \frac{1}{9}x$$

Solution:We want to find the petrol we need to travel 180 kilometres.

$$y = \frac{1}{9} \times 180 = 20$$

Interpretation : Since y = 20,We need 20 litres of petrol to travel 180 kilometres.

Example 2 : Suppose Sudhir has invested ₹ 15,000 at 8% simple interest per year. With the return from the investment, he wants to buy a washing machine that costs ₹ 19,000. For what period should he invest ₹ 15,000 so that he has enough money to buy a washing machine?

Solution:

Step 1 : Formulation of the problem : We have to find the number of years.

$$\text{Mathematical Description : Simple intterest I} = \frac{Pnr}{100}$$

P = Principal,

n = Number of years,

r % = Rate of interest

I = Interest earned

P=15,000; I=19,000-15,000=4,000; r=8%; we want n

$$\frac{15000 \times 8 \times n}{100} = 4000$$

$$n = \frac{4000}{150 \times 8} = \frac{10}{3} = 3\frac{1}{3}$$

$$Solution\ of\ the\ problem: n = 3\frac{1}{3}\ years = 3\ years\ and\ 4\ months$$

Interpretation : Sudhir can buy a washing machine after 3 years and 4 months.

Example 3 : A motorboat goes upstream on a river and covers the distance between two towns on the riverbank in six hours. It covers this distance downstream in five hours. If the speed of the stream is 2 km/h, find the speed of the boat in still water.

Solution :

Step 1 : Formulation : We have to find the speed of the boat in still water.

Mathematical Description : Let us write x for the speed of the boat, t for the time taken and y for the distance travelled. Then y = tx→ (1)

Let d be the distance between the two places.

While going upstream:

The speed of the boat upstream= speed of the boat – speed of the river = (x – 2) km/h,

It takes 6 hours to cover the distance between the towns upstream

d = 6(x – 2) →(2)

The speed of the boat downstream = (x + 2) km/h

The boat takes 5 hours to cover the same distance downstream.

d = 5(x + 2)→ (3)

From (2) and (3), we have 5(x + 2) = 6(x –2) →(4)

Step 2 : Finding the Solution

5(x + 2) = 6(x −2)

$5x + 10 = 6x - 12$

$6x - 5x = 10 + 12$

$x = 22$

Step 3 : Interpretation: Since x = 22, therefore the speed of the motorboat in still water is 22 km/h.

EXERCISE A 2.1

In each of the following problems, clearly state what the relevant and irrelevant factors are while going through Steps 1, 2 and 3 given above.

1. Suppose a company needs a computer for some period of time. The company can either hire a computer for ₹ 2,000 per month or buy one for ₹ 25,000. If the company has to use the computer for a long period, the company will pay such a high rent, that buying a computer will be cheaper. On the other hand, if the company has to use the computer for say, just one month, then hiring a computer will be cheaper. Find the number of months beyond which it will be cheaper to buy a computer.

Sol: The relevant factors are the time period for hiring a computer, and the two costs given to us. We assume that there is no significant change in the cost of purchasing or hiring the computer. So, we treat any such change as irrelevant. We also treat all brands and generations of computers as the same, i.e. these differences are also irrelevant.

The expense of hiring the computer for x months is ₹ 2000x. If this becomes more than the cost of purchasing a computer, we will be better off buying a computer. So, the equation is 2000 x = 25000→ (1)

Step 2 : Solution : Solving (1), x = 25000 2000 = 12.5 Step 3 : Interpretation : Since the cost of hiring a computer becomes more after 12.5 months, it is cheaper to buy a computer, if you have to use it for more than 12 months.

2. Suppose a car starts from a place A and travels at a speed of 40 km/h towards another place B. At the same instance, another car starts from B and travels towards A at a speed of 30 km/h. If the distance between A and B is 100 km, after how much time will the cars meet?

Sol: Step1 : Formulation : We will assume that cars travel at a constant speed. So, any change of speed will be treated as irrelevant. If the cars meet after x hours, the first car would have travelled a distance of 40x km from A and the second car would have travelled 30x km, so that it will be at a distance of (100 – 30x) km from A. So the equation will be 40x = 100 – 30x, i.e., 70x = 100.

Step 2 : Solution : Solving the equation, we get x = $\frac{100}{70}$.

Step 3 : Interpretation : $\frac{100}{70}$ is approximately 1.4 hours. So, the cars will meet after 1.4 hours.

3. The moon is about 3,84,000 km from the earth, and its path around the earth is nearly circular. Find the speed at which it orbits the earth, assuming that it orbits the earth in 24 hours. (Use π = 3.14)

Sol: Step1 : Formulation : The speed at which the moon orbits the earth is $\frac{\text{Length of the orbit}}{\text{Time taken}}$.

Step 2 : Solution : Since the orbit is nearly circular, the length is 2 × p × 384000 km = 2411520 km

The moon takes 24 hours to complete one orbit.

So, speed = $\frac{2411520}{24} =$ = 100480 km/hour.

Step 3 : Interpretation : The speed is 100480 km/h.

4. A family pays ₹ 1000 for electricity on an average in those months in which it does not use a water heater. In the months in which it uses a water heater, the average electricity bill is ₹ 1240. The cost of using the water heater is ₹ 8.00 per hour. Find the average number of hours the water heater is used in a day.

Sol: Formulation : An assumption is that the difference in the bill is only because of using the water heater. Let the average number of hours for which the water heater is used = x Difference per month due to using water heater = ₹ 1240 – ₹ 1000 = ₹ 240 Cost of using water heater for one hour = ₹ 8 So, the cost of using the water heater for 30 days = 8 × 30 × x Also, the cost of using the water heater for 30 days = Difference in bill due to using water heater So, 240x = 240 Solution : From this equation, we get x = 1. Interpretation : Since x = 1, the water heater is used for an average of 1 hour in a day.

A2.3 Some Mathematical Models

Example 4 : Suppose you have a room of length 6 m and breadth 5 m. You want to cover the floor of the room with square mosaic tiles of side 30 cm. How many tiles will you need? Solve this by constructing a mathematical model.

Solution: Formulation : We have to consider the area of the room and the area of a tile for solving the problem. The side of the tile is 0.3 m.

Since the length is 6 m, we can fit in 6 /0.3 = 20 tiles along the length of the room in one row

Since the breadth of the room is 5 metres, we have 5 / 0.3 = 16.67.

So, we can fit in 16 tiles in a column. Since 16 × 0.3 = 4.8, 5 – 4.8 = 0.2 metres along the breadth will not be covered by tiles.

Mathematical Description:

Total number of tiles required = (Number of tiles along the length × Number of tiles along the breadth) + Number of tiles along the uncovered area

(20 × 16) + 20 = 320 + 20 = 340.

Interpretation: We need 340 tiles to cover the floor.

Example 5 : In the year 2000, 191 member countries of the U.N. signed a declaration. In this declaration, the countries agreed to achieve certain development goals by the year 2015. These are called the millennium development goals. One of these goals is to promote gender equality. One indicator for deciding whether this goal has been achieved is the ratio of girls to boys in primary, secondary and tertiary education. India, as a signatory to the declaration, is committed to improve this ratio. The data for the percentage of girls who are enrolled in primary schools is given Table .

Year	Enrolment (in %)
1991-92	41.9
1992-93	42.6
1993-94	42.7
1994-95	42.9
1995-96	43.1
1996-97	43.2
1997-98	43.5
1998-99	43.5
1999-2000	43.6*
2000-01	43.7*
2001-02	44.1*

Solution:

Step 1 : Formulation: we will see how the enrolment grows after 1991 by comparing the number of years that has passed after 1991 and the enrolment. Let us take 1991 as the 0th year, and write 1 for 1992 since 1 year has passed in 1992 after 1991. Similarly, we will write 2 for 1993, 3 for 1994, etc. So, given table will now look like as Table given below. The increase in enrolment is also given in the following table :

Year	Enrolment (in %)	Increase
0	41.9	0
1	42.6	0.7
2	42.7	0.1
3	42.9	0.2
4	43.1	0.2
5	43.2	0.1
6	43.5	0.3
7	43.5	0
8	43.6	0.1
9	43.7	0.1
10	44.1	0.4

$$\text{The mean of the increasing values} = \frac{0.7+0.1+0.2+0.2+0.1+0.3+0+0.1+0.1+0.4}{10} = \frac{2.2}{10}$$
$$= 0.22$$

Let us assume that the enrolment steadily increases at the rate of 0.22 per cent.

Mathematical Description: We have assumed that the enrolment increases steadily at the rate of 0.22% per year.

So, the Enrolment Percentage (EP) in the first year = 41.9 + 0.22

EP in the second year = 41.9 + 2 × 0.22

EP in the third year = 41.9 + 3 × 0.22

So, the enrolment percentage in the n^{th} year = 41.9 + 0.22×n , for ≥ 1.→(1)

Now, we also have to find the number of years by which the enrolment will reach 50%.

So, we have to find the value of in the equation or formula 50=41.9+0.22n→(2)

Step 2: Solution:

$0.22n = 50 - 41.9 = 8.1$

$n = \frac{8.1}{0.22} = 36.8$

Step 3: Interpretation:

Since the number of years is an integral value, we will take the next higher integer, 37. So, the enrolment percentage will reach 50% in 1991 + 37 = 2028.

Step 4: Validation: Let us check if Formula (2) is in agreement with the reality. Let us find the values for the years we already know, using Formula (2), and compare it with the known values by finding the difference. The values are given in below Table.

Year	Enrolment (in %)	Values given by (2) (in %)	Difference (in %)
0	41.9	41.90	0
1	42.6	42.12	0.48
2	42.7	42.34	0.36
3	42.9	42.56	0.34
4	43.1	42.78	0.32
5	43.2	43.00	0.20
6	43.5	43.22	0.28
7	43.5	43.44	0.06
8	43.6	43.66	–0.06
9	43.7	43.88	–0.18
10	44.1	44.10	0.00

As you can see, some of the values given by Formula (2) are less than the actual values by about 0.3% or even by 0.5%. This can give rise to a difference of about 3 to 5 years since the increase per year is actually 1% to 2%. We may decide that this much of a difference is acceptable and stop here. In this case, (2) is our mathematical model. Suppose we decide that this error is quite large, and we have to improve this model. Then we have to go back to Step 1, the formulation, and change Equation (2). Let us do so. S 1 : R : We still assume that the values increase steadily by 0.22%, but we will now introduce a correction factor to reduce the error. For this, we find the mean of all the errors. This is

$$\frac{0 + 0.48 + 0.36 + 0.34 + 0.2 + 0.28 + 0.06 - 0.06 - 0.18 + 0}{10} = 0.18$$

We take the mean of the errors, and correct our formula by this value.

Revised Mathematical Description : Let us now add the mean of the errors to our formula for enrolment percentage given in (2). So, our corrected formula is:

Enrolment percentage in the n^{th} year = 41.9 + 0.22n + 0.18 = 42.08 + 0.22n , forn≥1→(3)

We will also modify Equation (2) appropriately. The new equation for is: 50=42.08+0.22n →(4)

Step 2 : Altered Solution : Solving Equation (4) for , we get

$$n = \frac{50 - 42.08}{0.22} = \frac{7.92}{0.22} = 36$$

Step 3 : Interpretation : Since = 36, the enrolment of girls in primary schools will reach 50% in the year 1991 + 36 = 2027.

Step 4 : Validation : Once again, let us compare the values got by using Formula (4) with the actual values. Table A2.5 gives the comparison.

Year	Enrolment (in %)	Values given by (2)	Difference between values	Values given by (4)	Difference between values
0	41.9	41.90	0	41.9	0
1	42.6	42.12	0.48	42.3	0.3
2	42.7	42.34	0.36	42.52	0.18
3	42.9	42.56	0.34	42.74	0.16
4	43.1	42.78	0.32	42.96	0.14
5	43.2	43.00	0.2	43.18	0.02
6	43.5	43.22	0.28	43.4	0.1
7	43.5	43.44	0.06	43.62	– 0.12
8	43.6	43.66	– 0.06	43.84	– 0.24
9	43.7	43.88	– 0.18	44.06	– 0.36
10	44.1	44.10	0	44.28	– 0.18

As you can see, many of the values that (4) gives are closer to the actual value than the values that (2) gives. The mean of the errors is 0 in this case.

So, Equation (4) is our mathematical description that gives a mathematical relationship between years and the percentage of enrolment of girls of the total enrolment. We have constructed a mathematical model that describes the growth.

EXERCISE A2.2

1. **We have given the timings of the gold medalists in the 400-metre race from the time the event was included in the Olympics, in the table below. Construct a mathematical model relating the years and timings. Use it to estimate the timing in the next Olympics.**

Year	Timing (in seconds)
1964	52.01
1968	52.03
1972	51.08
1976	49.28
1980	48.88
1984	48.83
1988	48.65
1992	48.83
1996	48.25
2000	49.11
2004	49.41

Sol: Let us first convert the problem into a mathematical problem.

Formulation: Let us take 1964 as 0^{th} year, and write 1^{st} for 1968, 2^{nd} for 1972 and so on. So given table will now look as table

Year	Timing (in seconds)
0	52.01
1	52.03
2	51.08
3	49.28
4	48.88
5	48.83
6	48.65
7	48.83
8	48.25
9	49.11
10	49.11

The reduction in timings of gold medalist in the following table.

Year	Timing (in seconds)	Difference
0	52.01	0
1	52.03	0.02
2	51.08	-0.95
3	49.28	-1.8
4	48.88	-0.4
5	48.83	-0.05
6	48.65	-0.18
7	48.83	0.18
8	48.25	-0.58
9	49.11	0.86
10	49.11	0.3
	Total	-2.6

$$Mean\ of\ differences = \frac{-2.6}{10} = -0.26$$

Let us assume that the timing steadily decreases at the rate of 0.26.

Mathematical Description:

$So, timing\ in\ the\ first\ year\ = 52.01 - 0.26$

$Timing\ in\ the\ second\ year\ = 52.01 - 0.26 \times 2$

$Timing\ in\ the\ n^{th}\ year\ (t) = 52.01 - 0.26 \times n$

Finding the solution:

The estimate timing in the next Olympics (n=11) is

$t = 52.01 - 0.26 \times 11 = 52.01 - 2.86 = 49.15\ sec$

EXERCISE A2.3

1. **How are the solving of word problems that you come across in textbooks different from the process of mathematical modelling?**

Sol: We have already mentioned that the formulation part could be very detailed in real-life situations. Also, we do not validate the answer in word problems. Apart from this word problem have a 'correct answer'. This need not be the case in real-life situations.

2. **Suppose you want to minimise the waiting time of vehicles at a traffic junction of four roads. Which of these factors are important and which are not? (i) Price of petrol. (ii)The rate at which the vehicles arrive in the four different roads. (iii) The proportion of slow-moving vehicles like cycles and rickshaws and fast moving vehicles like cars and motorcycles**

Sol: The important factors are (ii) and (iii). Here (i) is not an important factor although it can have an effect on the number of vehicles sold.