2. POLYNOMIALS(notes)

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1. An algebraic expression in which the variables involved have only non-negative integral (whole numbers) powers is called a polynomial.

Polynomials	Not polynomials
$2x$	$4x^{\frac{1}{2}}$
$\frac{1}{3}x-4$	$3x^2+4x^{-1}+5$
x^2-2x-1	$4+\frac{1}{x}$

2. If a polynomial contains only one variable then it is called polynomial in one variable.
   Ex: $2x+3;\quad 5x^2-6x+2;\quad 5y+6;\quad -6y^2+7y-5$
3. In the polynomial x^2+2x, the expressions x^2 and $2x$ are called the terms of the polynomial.
4. Each term of a polynomial has a coefficient. In $-x^3+4x^2+7x-2$
   The coefficient of $x^3=-1$
   The coefficient of $x^2=4$
   The coefficient of $x=7$
   constant term $=-2$
5. 2, –5, 7, etc. are examples of **constant polynomials**.
6. The constant polynomial 0 is called the **zero polynomial**.
7. If the variable in a polynomial is x, we may denote the polynomial by p(x), or q(x), or r(x), etc
8. The highest power of the variable in a polynomial as the degree of the polynomial.
   Example: i) $3x^2+7x+5$ → degree=2
   ii) $7x^3+5x^2+2x-6$ → degree=3

Types of polynomials according to degree

1. Constant polynomial: A polynomial of degree 0 is called constant polynomial.
   Ex: 5,-7,120,...
2. Linear polynomial: A polynomial of degree 1 is called a linear polynomial.
   Example: $3x+5, 7x-8, -9x, \ldots$
   The general form a linear polynomial in variable x is $ax+b$ $(a,b\in R, a\neq 0)$.
3. Quadratic polynomial : A polynomial of degree 2 is called a quadratic polynomial.
   Example: $x^2-5x+6, 2x^2-5, 7x^2, \ldots$..
   The general form a quadratic polynomial in variable x is ax^2+bx+c $(a,b,c\in R, a\neq 0)$.
4. Cubic polynomial : A polynomial of degree 3 is called a cubic polynomial.

Example: $5x^3 - 4x^2 + x - 1, 2x^3 - 3x + 5, -3x^3 - 10, \ldots\ldots$

The general form a cubic polynomial in variable x is $ax^3 + bx^2 + cx + d \ (a, b, c, d \in R, a \neq 0)$.

9. **The general form of n^{th} degree polynomial in one variable x**:

$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots\cdots + a_1 x + a_0$ is a polynomial of n^{th} degree ,

where $a_0, a_1, a_2, \ldots\ldots\ldots, a_{n-1}, a_n$ are real coefficients and $a_0 \neq 0$.

EXERCISE 2.1

1. **Which of the following expressions are polynomials in one variable and which are not? State reasons for your answer.**

$(i)\ 4x^2 - 3x + 7 \rightarrow$ polynomial in one variable x

$(ii)\ y^2 + \sqrt{2} \rightarrow$ polynomial in one variable y

$(iii)\ 3\sqrt{t} + t\sqrt{2} \rightarrow$ not a polynomial

$(iv)\ y + \frac{2}{y} \rightarrow$ not a polynomial

$(v)\ x^{10} + y^3 + t^{50} \rightarrow$ polynomial in three variables x, y and t

2. **Write the coefficient of x^2 in each of the following**

$(i)\ 2 + x^2 + x \rightarrow$ coefficient of $x^2 = 1$

$(ii)\ 2 - x^2 + x^3 \rightarrow$ coefficient of $x^2 = -1$

$(iii)\ \frac{\pi}{2}x^2 + x \rightarrow$ coefficient of $x^2 = \frac{\pi}{2}$

$(iv)\ \sqrt{2}x - 1 \rightarrow$ coefficient of $x^2 = 0$

3. **Give one example each of a binomial of degree 35, and of a monomial of degree 100**.

Sol: A binomial of degree 35 : $x^{35} + x^2$

A monomial of degree 100: $3x^{100}$

4. **Write the degree of each of the following polynomials:**

Polynomial	Degree
$(i)\ 5x^3 + 4x^2 7x$	3
$(ii)\ 4 - y^2$	2
$(iii) 5t - \sqrt{7}$	1
$(iv) 3$	0

5. **Classify the following as linear, quadratic and cubic polynomials:**

Sol: Linear polynomials: $(iv) 1 + x \quad (v) 3t$

Quadratic polynomials: $(i) x^2 + x \quad (iii)\ y + y^2 + 4 \quad (vi)\ r^2$

Cubic polynomials: $(ii)\ x - x^3 \quad (vii)\ 7x^3$

Example 2 : Find the value of each of the following polynomials at the indicated value of variables

(i) $\quad p(x) = 5x^2 - 3x + 7$ **at** $x = 1$

Sol: $p(1) = 5(1)^2 - 3(1) + 7 = 5 - 3 + 7 = 12 - 3 = 9$

(ii) $q(y) = 3y^2 - 4y + \sqrt{11}$ **at** $y = 2$

Sol: $q(2) = 3(2)^2 - 4(2) + \sqrt{11} = 12 - 8 + \sqrt{11} = 4 + \sqrt{11}$

(iii) $p(t) = 4t^4 + 5t^3 - t^2$ **at** $t = a$

Sol: $p(a) = 4a^4 + 5a^3 - a^2$

Zeroes of a Polynomial

1. A real number 'c' is a zero of a polynomial $p(x)$ if $p(c) = 0$. In this case, 'c' is also called a root of the polynomial equation $p(x) = 0$.
2. Every linear polynomial in one variable has a unique zero, a non-zero constant polynomial has no zero.
3. Every real number is a zero of the **zero polynomial**.

Linear Polynomial	Zero of the polynomial
$x + a$	$-a$
$x - a$	a
$ax + b$	$\frac{-b}{a}$
$ax - b$	$\frac{b}{a}$

Example 3 : Check whether –2 and 2 are zeroes of the polynomial x + 2.

Solu : Let $p(x) = x + 2$

Then $p(2) = 2 + 2 = 4, p(-2) = -2 + 2 = 0$

Therefore, –2 is a zero of the polynomial x + 2, but 2 is not.

Example 4 : Find a zero of the polynomial p(x) = 2x + 1.

Sol: Let $p(x) = 0$

$2x + 1 = 0$

$2x = -1$

$x = \frac{-1}{2}$

So, $\frac{-1}{2}$ *is* a zero of the polynomial $2x + 1$

Example 5 : Verify whether 2 and 0 are zeroes of the polynomial $x^2 - 2x$

Sol: *Let* $p(x) = x^2 - 2x$

$p(2) = (2)^2 - 2(2) = 4 - 4 = 0$

$p(0) = (0)^2 - 2(0) = 0 - 0 = 0$

Hence, 2 and 0 are both zeroes of the polynomial $x^2 - 2x$.

EXERCISE 2.2

1. Find the value of the polynomial $5x - 4x^2 + 3$ at (i) x = 0 (ii) x = −1 (iii) x = 2

Sol: $Let\ p(x) = 5x - 4x^2 + 3$

(i) $p(0) = 5(0) - 4(0)^2 + 3 = 0 - 0 + 3 = 3$

(ii) $p(-1) = 5(-1) - 4(-1)^2 + 3 = -5 - 4 + 3 = -9 + 3 = -6$

(iii) $p(2) = 5(2) - 4(2)^2 + 3 = 10 - 16 + 3 = 13 - 16 = -3$

2. Find p(0), p(1) and p(2) for each of the following polynomials:

$(\boldsymbol{i})\ \boldsymbol{p(y) = y^2 - y + 1}$

Sol: $p(0) = (0)^2 - 0 + 1 = 0 - 0 + 1 = 1$

$p(1) = (1)^2 - 1 + 1 = 1 - 1 + 1 = 1$

$p(2) = (2)^2 - 2 + 1 = 4 - 2 + 1 = 5 - 2 = 3$

$(\boldsymbol{ii})\ \boldsymbol{p(t) = 2 + t + 2t^2 - t^3}$

Sol: $p(0) = 2 + 0 + 2 \times (0)^2 - (0)^3 = 2 + 0 + 0 - 0 = 2$

$p(1) = 2 + 1 + 2 \times 1^2 - 1^3 = 2 + 1 + 2 - 1 = 4$

$p(2) = 2 + 2 + 2 \times 2^2 - 2^3 = 4 + 8 - 8 = 4$

$(\boldsymbol{iii})\ \boldsymbol{p(x) = x^3}$

Sol: $p(0) = 0^3 = 0$

$p(1) = 1^3 = 1$

$p(2) = 2^3 = 8$

$(\boldsymbol{iv})\ \boldsymbol{p(x) = (x - 1)(x + 1)}$

Sol: $p(0) = (0 - 1)(0 + 1) = (-1) \times 1 = -1$

$p(1) = (1 - 1)(1 + 1) = 0 \times 2 = 0$

$p(2) = (2 - 1)(2 + 1) = 1 \times 3 = 3$

3. Verify whether the following are zeroes of the polynomial, indicated against them

$(\boldsymbol{i})\ \boldsymbol{p(x) = 3x + 1;\ x = -\frac{1}{3}}$

$Sol\!: p\left(-\frac{1}{3}\right) = 3 \times \left(-\frac{1}{3}\right) + 1 = -1 + 1 = 0$

$p\left(-\frac{1}{3}\right) = 0$

$So, \left(-\frac{1}{3}\right)\ is\ a\ zero\ of\ the\ polynomial\ 3x + 1$

$(\boldsymbol{ii})\ \boldsymbol{p(x) = 5x - \pi;\ x = \frac{4}{5}}$

Sol: $p(\frac{4}{5}) = 5 \times \left(\frac{4}{5}\right) - \pi = 4 - \pi$

$p\left(\frac{4}{5}\right) \neq 0$

So $\frac{4}{5}$ is not a zero of the polynomial $5x - \pi$.

(iii) $\boldsymbol{p(x) = x^2 - 1;\ x = 1, -1}$

Sol: $p(1) = 1^2 - 1 = 1 - 1 = 0$

$p(-1) = (-1)^2 - 1 = 1 - 1 = 0$

$p(1) = 0\ and\ p(-1) = 0$

So, $1, -1$ *are the zeroes of the polynomial* $x^2 - 1$.

(iv) $\boldsymbol{p(x) = (x + 1)(x - 2), x = -1, 2}$

sol: $p(-1) = (-1 + 1)(-1 - 2) = 0 \times (-3) = 0$

$p(2) = (2 + 1)(2 - 2) = 3 \times 0 = 0$

So, $-1,2$ *are the zeroes of the polynomial* $(x + 1)(x - 2)$

(v) $\boldsymbol{p(x) = x^2 ;\ x = 0}$

Sol: $p(0) = 0^2 = 0$

So, 0 is a zero of the polynomial x^2

(vi) $\boldsymbol{p(x) = lx + m, x = -\frac{m}{l}}$

Sol: $p(x) = lx + m$

$p\left(-\frac{m}{l}\right) = l\left(-\frac{m}{l}\right) + m = -m + m = 0$

(vii) $\boldsymbol{p(x) = 3x^2 - 1, x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}}$

Sol: $p\left(-\frac{1}{\sqrt{3}}\right) = 3\left(-\frac{1}{\sqrt{3}}\right)^2 - 1 = 3 \times \frac{1}{3} - 1 = 1 - 1 = 0$

$p\left(\frac{2}{\sqrt{3}}\right) = 3\left(\frac{2}{\sqrt{3}}\right)^2 - 1 = 3 \times \frac{4}{3} - 1 = 4 - 1 = 3$

$-\frac{1}{\sqrt{3}}$ is the zero of the polynomial $3x^2 - 1$, but $\frac{2}{\sqrt{3}}$ is not.

(viii) $\boldsymbol{p(x) = 2x + 1, x = \frac{1}{2}}$

Sol: $p\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) + 1\ \ 2 + 1 = 3$

$\frac{1}{2}$ is not a zero of the polynomial $2x + 1$

4. Find the zero of the polynomial in each of the following cases

(i) $\boldsymbol{p(x) = x + 5}$

Sol: $Let\ p(x) = 0$

$x + 5 = 0$

$x = -5$

$\therefore$ -5 is the zero of the polynomial $x + 5$

$(\boldsymbol{ii})$ $\boldsymbol{p(x) = x - 5}$

Sol: $Let\ p(x) = 0$

$x - 5 = 0$

$x = 5$

$\therefore$ $5\ is\ the\ zero\ of\ the\ polynomial\ x - 5$

$(\boldsymbol{iii})$ $\boldsymbol{p(x) = 2x + 5}$

Sol: $Let\ p(x) = 0$

$2x + 5 = 0$

$2x = -5 \Rightarrow x = \frac{-5}{2}$

$\therefore \frac{-5}{2}\ is\ the\ zero\ of\ the\ polynomial\ 2x + 5$

$(\boldsymbol{vi})$ $\boldsymbol{p(x) = 3x - 2}$

Sol: $Let\ p(x) = 0$

$3x - 2 = 0$

$3x = 2 \Rightarrow x = \frac{2}{3}$

$\therefore \frac{2}{3}\ is\ the\ zero\ of\ the\ polynomial\ 3x - 2$

$(\boldsymbol{v})$ $\boldsymbol{p(x) = 3x}$

Sol: $Let\ p(x) = 0$

$3x = 0$

$x = 0$

$\therefore$ $0\ is\ the\ zero\ of\ the\ polynomial\ 3x$

$(\boldsymbol{vi})$ $\boldsymbol{p(x) = ax\ , a \neq 0}$

Sol: $Let\ p(x) = 0$

$ax = 0$

$x = 0$

$\therefore$ $0\ is\ the\ zero\ of\ the\ polynomial\ ax$

$(\boldsymbol{vii})$ $\boldsymbol{p(x) = cx + d, c \neq 0, c, d\ are\ real\ numbers.}$

Sol: $Let\ p(x) = 0$

$cx + d = 0$

$cx = -d \Rightarrow x = \frac{-d}{c}$

$\therefore \frac{-d}{c}$ *is the zero of the polynomial* $cx + d$

Remainder Theorem: Let $p(x)$ be any polynomial of degree greater than or equal to one and let 'a' be any real number. If $p(x)$ is divided by the linear polynomial $(x - a)$, then the remainder is $p(a)$.

Factor Theorem : $p(x)$ is a polynomial of degree n $\geq$ 1 and 'a' is any real number

(i) $If\ p(a) = 0\ then\ (x - a\ ) is\ a\ factor\ of\ p(x)$. and

(ii) $If\ (x - a)\ is\ a\ factor\ of\ polynomial\ p(x)\ then\ p(a) = 0.$

$Example\ 6 :\ Examine\ whether\ x + 2\ is\ a\ factor\ of\ x^3 + 3x^2 + 5x + 6\ and\ of\ 2x + 4.$

Sol: Let $p(x) = x^3 + 3x^2 + 5x + 6$

$p(-2) = (-2)^3 + 3(-2)^2 + 5(-2) + 6$

$= -8 + 12 - 10 + 6 = 18 - 18 = 0$

$\therefore x + 2\ is\ a\ factor\ of\ x^3 + 3x^2 + 5x + 6$

$since\ 2x + 4 = 2(x + 2)\ but\ 2\ is\ not\ a\ factor\ of\ p(x)$

$So, 2x + 4\ is\ not\ a\ factor\ of\ p(x)$

Example 7 : Find the value of k, $if\ x - 1$ is a factor of $4x^3 + 3x^2 - 4x + k$

Sol: $Let\ p(x) = 4x^3 + 3x^2 - 4x + k$

$If\ x - 1\ is\ a\ factor\ of\ p(x) then\ p(1) = 0$

$4(1)^3 + 3(1)^2 - 4(1) + k = 0$

$4 + 3 - 4 + k = 0$

$3 + k = 0$

$k = -3$

$Factorisation\ of\ the\ polynomial\ ax^2 + bx + c\ by\ splitting\ the\ middle\ term$.

Let its factors be $(px + q)$ and $(rx + s)$.

$ax^2 + bx + c = (px + q)(rx + s)$

$ax^2 + bx + c = prx^2 + (ps + qr)x + qs$

$ps \times qr = a \times c\ and\ \ ps + qr = b$

Example 8 : Factorise $6x^2 + 17x + 5$ by splitting the middle term, and by using the Factor Theorem

Sol: $6x^2 + 17x + 5 = 6x^2 + 2x + 15x + 5$

$= 2x(3x + 1) + 5(3x + 1)$

$= (3x + 1)(2x + 5)$

Example 9 : Factorise $y^2 - 5y + 6$ by using the Factor Theorem.

Sol: $p(y) = y^2 - 5y + 6$

$p(1) = (1)^2 - 5(1) + 6 = 1 - 5 + 6 = 7 - 5 = 2$

$p(2) = (2)^2 - 5(2) + 6 = 4 - 10 + 6 = 10 - 10 = 0$

$(y - 2)$ *is a factor of* $p(y)$

$p(3) = (3)^2 - 5(3) + 6 = 9 - 15 + 6 = 15 - 15 = 0$

$(y - 3)$ *is a factor of* $p(y)$

$\therefore y^2 - 5y + 6 = (y - 2)(y - 3)$

Example 10 : Factorise $\boldsymbol{x^3 - 23x^2 + 142x - 120}$.

Sol: $p(x) = x^3 - 23x^2 + 142x - 120$

$p(1) = (1)^3 - 23(1)^2 + 142(1) - 120 = 1 - 23 + 142 - 120 = 143 - 143 = 0$

$\therefore (x - 1)$ *is a factor of* $p(x)$

$x^3 - 23x^2 + 142x - 120 = x^3 - x^2 - 22x^2 + 22x + 120x - 120$

$= x^2(x - 1) - 22x(x - 1) + 120(x - 1)$

$= (x - 1)(x^2 - 22x + 120)$

$(x^2 - 22x + 120) = (x^2 - 12x - 10x + 120)$

$= x(x - 12) - 10(x - 12)$

$= (x - 12)(x - 10)$

$x^3 - 23x^2 + 142x - 120 = (x - 1)(x - 12)(x - 10)$

EXERCISE 2.3

1. Determine which of the following polynomials has (x + 1) a factor

(*i*) $\mathbf{x^3 + x^2 + x + 1}$

Sol: $p(x) = x^3 + x^2 + x + 1$

$p(-1) = (-1)^3 + (-1)^2 + (-1) + 1$

$= -1 + 1 - 1 + 1 = 2 - 2 = 0$

$(x + 1)$ is a factor *of* $x^3 + x^2 + x + 1$

(*ii*) $\mathbf{x^4 + x^3 + x^2 + x + 1}$

Sol: $p(x) = x^4 + x^3 + x^2 + x + 1$

$p(-1) = (-1)^4 + (-1)^3 + (-1)^2 + (-1) + 1$

$= 1 - 1 + 1 - 1 + 1 = 3 - 2 = 1$

$(x + 1)$ is not a factor *of* $x^4 + x^3 + x^2 + x + 1$

(*iii*) $\boldsymbol{x^4 + 3x^3 + 3x^2 + x + 1}$

Sol: $p(x) = x^4 + 3x^3 + 3x^2 + x + 1$

$p(-1) = (-1)^4 + 3 \times (-1)^3 + 3 \times (-1)^2 + (-1) + 1$

$= 1 - 3 + 3 - 1 + 1 = 1$

$(x + 1)$ is not a factor *of* $x^4 + 3x^3 + 3x^2 + x + 1$

(*iv*) $\mathbf{x^3 - x^2 - (2 + \sqrt{2})\,x + \sqrt{2}}$

$Sol: \ p(x) = x^3 - x^2 - (2+\sqrt{2})\,x + \sqrt{2}$

$p(-1) = (-1)^3 - (-1)^2 - (2+\sqrt{2}) \times (-1) + \sqrt{2}$

$= -1 - 1 + 2 - \sqrt{2} + \sqrt{2} = 0$

$(x + 1)$is a factor of $x^3 - x^2 - (2+\sqrt{2})\,x + \sqrt{2}$

2. Use the Factor Theorem to determine whether g(x) is a factor of p(x) in each of the following cases:

$(\boldsymbol{i})\ \boldsymbol{p(x) = 2x^3 + x^2 - 2x - 1, g(x) = x + 1}$

Sol: $p(x) = 2x^3 + x^2 - 2x - 1$

$p(-1) = 2(-1)^3 + (-1)^2 - 2(-1) - 1$

$= -2 + 1 + 2 - 1 = 0$

$\therefore\ g(x)\ is\ a\ factor\ of\ p(x)$

$(ii)\ p(x) = x^3 + 3x^2 + 3x + 1, g(x) = x + 2$

Sol:$p(x) = x^3 + 3x^2 + 3x + 1$

$p(-2) = (-2)^3 + 3(-2)^2 + 3(-2) + 1$

$= -8 + 12 - 6 + 1$

$= 14 - 14 = 0$

$\therefore\ g(x)\ is\ a\ factor\ of\ p(x)$

$(iii) p(x) = x^3 - 4x^2 + x + 6, g(x) = x - 3$

Sol: $p(x) = x^3 - 4x^2 + x + 6$

$p(3) = 3^3 - 4 \times 3^2 + 2 + 6$

$= 27 - 36 + 8$

$= 36 - 36 = 0$

$\therefore\ g(x)\ is\ a\ factor\ of p(x)$

3. Find the value of k, if x – 1 is a factor of p(x) in each of the following cases:

$(\boldsymbol{i})\ \boldsymbol{p(x) = x^2 + x + k}$

If $x -$ 1 is a factor of p(x)then p(1) = 0

$1^2 + 1 + k = 0$

$2 + k = 0$

$k = -2$

$(ii)\ \boldsymbol{p(x) = 2x^2 + kx + 2}$

If $x - 1$ is a factor of $p(x)$ then $p(1) = 0$

$2 \times 1^2 + k \times 1 + 2 = 0$

$2 + k + 2 = 0$

$k + 4 = 0$

$k = -4$

$(iii)\ \boldsymbol{p(x) = kx^2 - \sqrt{2}\,x + 1}$

If $x - 1$ is a factor of $p(x)$ then $p(1) = 0$

$k \times (1)^2 - \sqrt{2} \times 1 + 1 = 0$

$k - \sqrt{2} + 1 = 0$

$k = \sqrt{2} - 1$

$(iv)\ p(x) = kx^2 - 3x + k$

If $x - 1$ is a factor of $p(x)$ then $p(1) = 0$

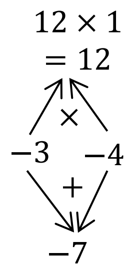
$k(1)^2 - 3(1) + k = 0$

$k - 3 + k = 0$

$2k - 3 = 0$

$k = \frac{3}{2}$

4. Factorise :

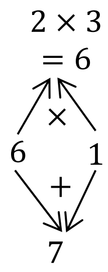


$(i)\ \boldsymbol{12x^2 - 7x + 1}$

$Sol: 12x^2 - 7x + 1 = 12x^2 - 3x - 4x + 1$

$= 3x(4x - 1) - 1(4x - 1)$

$= (4x - 1)(3x - 1)$

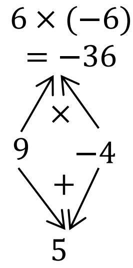


$(ii)\ \boldsymbol{2x^2 + 7x + 3}$

$Sol: 2x^2 + 7x + 3 = 2x^2 + 6x + x + 3$

$= 2x(x + 3) + 1(x + 3)$

$= (x + 3)(2x + 1)$



$(iii)\ \boldsymbol{6x^2 + 5x - 6}$

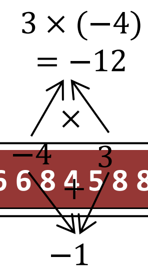
$Sol: 6x^2 + 5x - 6 = 6x^2 + 9x - 4x - 6$

$= 3x(2x + 3) - 2(2x + 3)$

$= (2x + 3)(3x - 2)$

$(iv)\ \boldsymbol{3x^2 - x - 4}$

$Sol: 3x^2 - x - 4 = 3x^2 - 4x + 3x - 4$



$= x(3x-4)+1(3x-4)$

$= (3x-4)(x+1)$

5. Factorise :

(i) $x^3 - 2x^2 - x + 2$

Sol: Let $p(x) = x^3 - 2x^2 - x + 2$

$p(x) = x^3 - 2x^2 - x + 2$

$p(1) = 1^3 - 2\times 1^2 - 1 + 2$

$= 1-2-1+2 = 0$

So $(x-1)$ is a factor of $p(x)$

$x^3 - 2x^2 - x + 2$

$= x^3 - x^2 - x^2 + x - 2x + 2$

$= x^2(x-1) - x(x-1) - 2(x-1)$

$= (x-1)(x^2-x-2)$

$= (x-1)(x+1)(x-2)$

$x^2 - x - 2$ $= x^2 - 2x + x - 2$ $= x(x-2)+1(x-2)$	$-1\times 2 = -2$ $-1+2 = 1$

(ii) $x^3 - 3x^2 - 9x - 5$

Sol: $p(x) = x^3 - 3x^2 - 9x - 5$

$p(-1) = (-1)^3 - 3\times(-1)^2 - 9\times(-1) - 5$

$= -1-3+9-5$

$= 9-9 = 0$

So $(x+1)$ is a factor of $p(x)$

$x^3 - 3x^2 - 9x - 5$

$= x^3 + x^2 - 4x^2 - 4x - 5x - 5$

$= x^2(x+1) - 4x(x+1) - 5(x+1)$

$= (x+1)(x^2-4x-5)$

$= (x+1)(x+1)(x-5)$

$= (x+1)^2(x-5)$

$x^2 - 4x - 5$ $= x^2 - 5x + x - 5$ $= x(x-5)+1(x-5)$	$-5\times 1 = -5$ $-5+1 = -4$

(iii) $x^3 + 13x^2 + 32x + 20$

Sol: $p(x) = x^3 + 13x^2 + 32x + 20$

$p(-1) = (-1)^3 + 13\times(-1)^2 + 32\times(-1) + 20$

$= -1+13-32+20$

$= 33-33 = 0$

So $(x+1)$ is a factor of $p(x)$

$x^3 + 13x^2 + 32x + 20$

$= x^3 + x^2 + 12x^2 + 12x + 20x + 20$

$= x^2(x+1) + 12x(x+1) + 20(x+1)$

$= (x+1)(x^2 + 12x + 20)$

$= (x+1)(x+2)(x+10)$

$x^2 + 12x + 20$	
$= x^2 + 2x + 10x + 20$	$2 \times 10 = 20$
$= x(x+2) + 10(x+2)$	$2 + 10 = 12$

$(iv)\mathbf{2y^3 + y^2 - 2y - 1}$

Sol: $p(y) = y^3 + y^2 - y - 1$

$p(-1) = (-1)^3 + (-1)^2 - (-1) - 1$

$= -1 + 1 + 1 - 1$

$= 2 - 2 = 0$

So $(y+1)$ is a factor of $p(y)$

$y^3 + y^2 - y - 1 = y^3 + y^2 - y - 1$

$= y^2(y+1) - 1(y+1)$

$= (y+1)(y^2 - 1)$

$= (y+1)(y+1)(y-1)$

Algebraic Identities

(i) $(x + y)^2 \equiv x^2 + 2xy + y^2$

(ii) $(x - y)^2 \equiv x^2 - 2xy + y^2$

(iii) $(x + y)(x - y) \equiv x^2 - y^2$

(iv) $(x + a)(x + b) \equiv x^2 + (a + b)x + ab.$

Example 11 : Find the following products using appropriate identities:

$(\boldsymbol{i})(\boldsymbol{x + 3})(\boldsymbol{x + 3})$

Sol: $(x + y)^2 = x^2 + 2xy + y^2$

$(x + 3)(x + 3) = (x + 3)^2 = x^2 + 2 \times x \times 3 + 3^2$

$= x^2 + 6x + 9$

$(\boldsymbol{ii})(\boldsymbol{x - 3})(\boldsymbol{x + 5})$

Sol: $(x + a)(x + b) = x^2 + (a + b)x + ab$

$(x - 3)(x + 5) = x^2 + (-3 + 5)x + (-3) \times 5$

$= x^2 + 2x - 15$

Example 12 : Evaluate 105 × 106 without multiplying directly

Sol: $(x + a)(x + b) \equiv x^2 + (a + b)x + ab$

$105 \times 106 = (100 + 5)(100 + 6)$

$= (100)^2 + (5 + 6) \times 100 + 5 \times 6$

$= 10000 + 1100 + 30 = 11130$

Example 13 : Factorise:

$(i)\mathbf{49a^2 + 70ab + 25b^2}$

Sol: $x^2 + 2xy + y^2 = (x + y)^2$

$49a^2 + 70ab + 25b^2 = (7a)^2 + 2 \times 7a \times 5b + (5b)^2$

$= (7a + 5b)^2 = (7a + 5b)(7a + 5b)$

$(ii)\ \dfrac{25}{4}x^2 - \dfrac{y^2}{9}$

Sol: $x^2 - y^2 = (x + y)(x - y)$

$$\frac{25}{4}x^2 - \frac{y^2}{9} = \left(\frac{5}{2}x\right)^2 - \left(\frac{y}{3}\right)^2 = \left(\frac{5}{2}x + \frac{y}{3}\right)\left(\frac{5}{2}x - \frac{y}{3}\right)$$

Identity V : $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

Sol: $(x + y + z)^2 = [(x + y) + z]^2 = (x + y)^2 + 2(x + y)z + z^2$

$= x^2 + 2xy + y^2 + 2xz + 2yz + z^2$

$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

Example 14 : Write $(3a + 4b + 5c)^2$ in expanded form

Sol: $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

$(3a + 4b + 5c)^2 = (3a)^2 + (4b)^2 + (5c)^2 + 2(3a)(4b) + 2(4b)(5c) + 2(3a)(5c)$

$= 9a^2 + 16b^2 + 25c^2 + 24ab + 40bc + 30ac$

***Example* 16 : *Factorise* $4x^2 + y^2 + z^2 - 4xy - 2yz + 4xz$**

Sol: $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = (x + y + z)^2$

$4x^2 + y^2 + z^2 - 4xy - 2yz + 4xz$

$= (2x)^2 + (-y)^2 + (z)^2 + 2(2x)(-y) + 2(-y)(z) + 2(2x)(z)$

$= (2x - y + z)^2 = (2x - y + z)(2x - y + z)$

Identity VI : $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 + 3xy(x + y)$

Sol: $(x + y)^3 = (x + y)(x + y)^2 = (x + y)(x^2 + 2xy + y^2)$

$= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3 = x^3 + y^3 + 3x^2y + 3xy^2 = x^3 + y^3 + 3xy(x + y)$

Identity VII : $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3 = x^3 - y^3 - 3xy(x - y)$

Sol: $(x - y)^3 = (x - y)(x - y)^2 = (x - y)(x^2 - 2xy + y^2)$

$= x^3 - 2x^2y + xy^2 - x^2y + 2xy^2 - y^3 = x^3 - y^3 - 3x^2y + 3xy^2 = x^3 - y^3 - 3xy(x - y)$

Example 17 : Write the following cubes in the expanded form:

$(i)(3a + 4b)^3$

Sol: $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$

$(3a + 4b)^3 = (3a)^3 + (4b)^3 + 3(3a)(4b)(3a + 4b)$

$= 27a^3 + 64b^4 + 36ab(3a + 4b) = 27a^3 + 64b^4 + 108a^2b + 144ab^2$

$(ii)\ (5p - 3q)^3$

Sol: $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$

$(5p - 3q)^3 = (5p)^3 - (3q)^3 - 3(5p)(3q)(5p - 3q)$

$= 125p^3 - 27q^3 - 45pq(5p - 3q)$

$= 125p^3 - 27q^3 - 225p^2q + 135pq^2$

Example 18 : Evaluate each of the following using suitable identities:

$(\boldsymbol{i})\ (\mathbf{104})^3$

$Sol: (x + y)^3 = x^3 + y^3 + 3xy(x + y)$

$(104)^3 = (100 + 4)^3 = (100)^3 + (4)^3 + 3(100)(4)(100 + 4)$

$= 1000000 + 64 + 1200 \times 104$

$= 1000000 + 64 + 124800 = 1124864$

$(\boldsymbol{ii})(\mathbf{999})^3$

$Sol: (x - y)^3 = x^3 - y^3 - 3xy(x - y)$

$(999)^3 = (1000 - 1)^3 = (100)^3 - (1)^3 - 3(100)(1)(1000 - 1)$

$= 1000000 - 1 - 300 \times 999$

$= 1000000 - 1 - 2997000 = 997002999$

Example 19 : $\boldsymbol{Factorise}\ \mathbf{8}\boldsymbol{x}^3 + \mathbf{27}\boldsymbol{y}^3 + \mathbf{36}\boldsymbol{x}^2\boldsymbol{y} + \mathbf{54}\boldsymbol{xy}^2$

$Sol:\ x^3 + y^3 + 3x^2y + 3xy^2 = (x + y)^3$

$8x^3 + 27y^3 + 36x^2y + 54xy^2 = (2x)^3 + (3y)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 = (2x + 3y)^3$

Identity VIII : $\boldsymbol{x}^3 + \boldsymbol{y}^3 + \boldsymbol{z}^3 - 3\boldsymbol{xyz} = (\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z})(\boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2 - \boldsymbol{xy} - \boldsymbol{yz} - \boldsymbol{xz})$

$\boldsymbol{Example}\ \mathbf{20} : \boldsymbol{Factorise} :\ \mathbf{8}\boldsymbol{x}^3 + \boldsymbol{y}^3 + \mathbf{27}\boldsymbol{z}^3 - \mathbf{18}\boldsymbol{xyz}$

$Sol:\ x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz)$

$8x^3 + y^3 + 27z^3 - 18xyz = (2x)^3 + (y)^3 + (3z)^3 - 3(2x)(y)(3z)$

$= (2x + y + 3z)[(2x)^2 + (y)^2 + (3z)^2 - (2x)(y) - (y)(3z) - (2x)(3z)]$

$= (2x + y + 3z)(4x^2 + y^2 + 9z^2 - 2xy - 3yz - 6xz)$

EXERCISE 2.4

1. Use suitable identities to find the following products

$(\boldsymbol{i})(\boldsymbol{x} + \mathbf{4})(\boldsymbol{x} + \mathbf{10})$

$Sol: (x + a)(x + b) \equiv x^2 + (a + b)x + ab\ ; a = 4, b = 10$

$(x + 4)(x + 10) = x^2 + (4 + 10)x + 4 \times 10 = x^2 + 14x + 40$

$(\boldsymbol{ii})(\boldsymbol{x} + \mathbf{8})(\boldsymbol{x} - \mathbf{10})$

$Sol: (x + a)(x + b) \equiv x^2 + (a + b)x + ab; a = 8, b = -10$

$(x + 8)(x - 10) = x^2 + (8 - 10)x + 8 \times (-10) = x^2 - 2x - 80$

$(\boldsymbol{iii})(\mathbf{3x} + \mathbf{4})(\mathbf{3x} - \mathbf{5})$

$Sol: (x + a)(x + b) \equiv x^2 + (a + b)x + ab;\ x = 3x, a = 4, b = -5$

$(3x + 4)(3x - 5) = (3x)^2 + (4 - 5)(3x) + 4 \times (-5)$

$= 9x^2 - 3x - 20$

$(iv)\left(y^2+\frac{3}{2}\right)\left(y^2-\frac{3}{2}\right)$

$Sol: (a+b)(a-b) = a^2 - b^2;\ \ a = y^2, b = \frac{3}{2}$

$$\left(y^2+\frac{3}{2}\right)\left(y^2-\frac{3}{2}\right) = (y^2)^2 - \left(\frac{3}{2}\right)^2 = y^4 - \frac{9}{4}$$

$(v)(3 - 2x)(3 + 2x)$

$Sol: (a+b)(a-b) = a^2 - b^2; a = 3, b = 2x$

$$(3 - 2x)(3 + 2x) = (3)^2 - (2x)^2 = 9 - 4x^2$$

2. Evaluate the following products without multiplying directly

$(i) 103 \times 107$

$Sol: (x + a)(x + b) \equiv x^2 + (a + b)x + ab; x = 100, a = 3, b = 7$

$$103 \times 107 = (100+3)(100+7) = (100)^2 + (3 + 7)(100) + 3 \times 7$$
$$= 10000 + 1000 + 21 = 11021$$

$(ii)\ 95 \times 96$

$Sol: (x + a)(x + b) \equiv x^2 + (a + b)x + ab; x = 90, a = 5, b = 6$

$$95 \times 96 = (90+5)(90+6) = (90)^2 + (5 + 6)(90) + 5 \times 6$$
$$= 8100 + 990 + 30 = 9120$$

$(iii) 104 \times 96$

$Sol: (a+b)(a-b) = a^2 - b^2; a = 100, b = 4$

$$104 \times 96 = (100+4)(100-4) = (100)^2 - (4)^2 = 10000 - 16 = 9984$$

3. Factorise the following using appropriate identities

$(i)\ 9x^2 + 6xy + y^2$

$Sol: a^2 + 2ab + b^2 = (a+b)^2;\ \ a = 3x, b = y$

$$9x^2 + 6xy + y^2 = (3x)^2 + 2(3x)(y) + (y)^2 = (3x+y)^2$$

$(ii)\ 4y^2 - 4y + 1$

$Sol: a^2 - 2ab + b^2 = (a-b)^2;\ \ a = 2y, b = 1$

$$4y^2 - 4y + 1 = (2y)^2 + 2(2y)(1) + (1)^2 = (2y+1)^2$$

$(iii)\ x^2 - \frac{y^2}{100}$

$Sol: a^2 - b^2 = (a+b)(a-b); a = x, b = \frac{y}{10}$

$$x^2 - \frac{y^2}{100} = x^2 - \left(\frac{y}{10}\right)^2 = \left(x+\frac{y}{10}\right)\left(x-\frac{y}{10}\right)$$

4. Expand each of the following, using suitable identities:

$(i)(x + 2y + 4z)^2$

$Sol: (a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$

$(x+2y+4z)^2=(x)^2+(2y)^2+(4z)^2+2(x)(2y)+2(2y)(4z)+2(x)(4z)$

$=x^2+4y^2+16z^2+4xy+16yz+8xz$

$(\boldsymbol{ii})(\mathbf{2x-y+z})^2$

$Sol: (a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ac$

$(\mathbf{2x-y+z})^2=(2x)^2+(-y)^2+(z)^2+2(2x)(-y)+2(-y)(z)+2(2x)(z)$

$=4x^2+y^2+z^2-4xy-2yz+4xz$

$(\boldsymbol{iii})(\mathbf{-2x+3y+2z})^2$

$Sol: (a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ac$

$(-2x+3y+2z)^2=(-2x)^2+(3y)^2+(2z)^2+2(-2x)(3y)+2(3y)(2z)+2(-2x)(2z)$

$=4x^2+9y^2+4z^2-12xy+12yz-8xz$

$(\boldsymbol{iv})(\mathbf{3a-7b-c})^2$

$Sol: (x+y+z)^2=x^2+y^2+z^2+2xy+2yz+2xz$

$(3a-7b-c)^2=(3a)^2+(-7b)^2+(-c)^2+2(3a)(-7b)+2(-7b)(-c)+2(3a)(-c)$

$=9a^2+49b^2+c^2-42ab+14bc-6ac$

$(\boldsymbol{v})(\mathbf{-2x+5y-3z})^2$

$Sol: (a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ac$

$(-2x+5y-3z)^2=(-2x)^2+(5y)^2+(-3z)^2+2(-2x)(5y)+2(5y)(-3z)+2(-2x)(-3z)$

$=4x^2+25y^2+9z^2-20xy-30yz+12xz$

$(\boldsymbol{vi})\left[\mathbf{\frac{1}{4}a-\frac{1}{2}b+1}\right]^2$

$Sol: (x+y+z)^2=x^2+y^2+z^2+2xy+2yz+2xz$

$\left(\frac{1}{4}a-\frac{1}{2}b+1\right)^2=\left(\frac{1}{4}a\right)^2+\left(-\frac{1}{2}b\right)^2+(1)^2+2\left(\frac{1}{4}a\right)\left(-\frac{1}{2}b\right)+2\left(-\frac{1}{2}b\right)(1)+2\left(\frac{1}{4}a\right)(1)$

$=\frac{1}{16}a^2+\frac{1}{4}b^2+1-\frac{1}{4}ab-b+\frac{1}{2}a=\frac{a^2}{16}+\frac{b^2}{4}+1-\frac{ab}{4}-b+\frac{a}{2}$

5. Factorise:

$(\boldsymbol{i})\mathbf{4x^2+9y^2+16z^2+12xy-24yz-16xz}$

$Sol: a^2+b^2+c^2+2ab+2bc+2ac=(a+b+c)^2$

$4x^2+9y^2+16z^2+12xy-24yz-16xz$

$=(2x)^2+(3y)^2+(-4z)^2+2(2x)(3y)+2(3y)(-4z)+2(2x)(-4z)$

$=(2x+3y-4z)^2$

$(\boldsymbol{ii})\mathbf{2x^2+y^2+8z^2-2\sqrt{2}xy+4\sqrt{2}yz-8xz}$

$Sol: a^2+b^2+c^2+2ab+2bc+2ac=(a+b+c)^2$

$2x^2+y^2+8z^2-2\sqrt{2}xy+4\sqrt{2}yz-8xz$

$=\left(-\sqrt{2}x\right)^2+(y)^2+\left(2\sqrt{2}z\right)^2+2\left(-\sqrt{2}x\right)(y)+2(y)\left(2\sqrt{2}z\right)+2\left(-\sqrt{2}x\right)\left(2\sqrt{2}z\right)$

$= \left(-\sqrt{2}x + y2\sqrt{2}z\right)^2$

6. Write the following cubes in expanded form

$(\boldsymbol{i})(\boldsymbol{2x+1})^{\boldsymbol{3}}$

$Sol: (a + b)^3 = a^3 + b^3 + 3ab\,(a + b)$

$(2x+1)^3 = (2x)^3 + (1)^3 + 3(2x)(1)(2x+1)$

$= 8x^3 + 1 + 6x(2x+1)$

$= 8x^3 + 1 + 12x^2 + 6x$

$= 8x^3 + 12x^2 + 6x + 1$

$(ii)(2a-3b)^3$

$Sol: (x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$

$(2a-3b)^3 = (2a)^3 - (3b)^3 - 3(2a)(3b)(2a-3b)$

$= 8a^3 - 27b^3 - 18ab(2a-3b)$

$= 8a^3 - 27b^3 - 36a^2b + 54ab^2$

$(\boldsymbol{iii})\left[\frac{\boldsymbol{3}}{\boldsymbol{2}}\boldsymbol{x}+\boldsymbol{1}\right]^{\boldsymbol{3}}$

$Sol: (a + b)^3 = a^3 + b^3 + 3ab\,(a + b)$

$\left[\frac{3}{2}x+1\right]^3 = \left(\frac{3}{2}x\right)^3 + (1)^3 + 3\left(\frac{3}{2}x\right)(1)\left(\frac{3}{2}x+1\right)$

$= \frac{27}{8}x^3 + 1 + \frac{9}{2}x\left(\frac{3}{2}x+1\right)$

$= \frac{9}{4}x^2 + 1 + \frac{9}{4}x^2 + \frac{9}{2}x$

$(\boldsymbol{iv})\left[\boldsymbol{x}-\frac{\boldsymbol{2}}{\boldsymbol{3}}\boldsymbol{y}\right]^{\boldsymbol{3}}$

$Sol: (a - b)^3 = a^3 - b^3 - 3ab\,(a - b)$

$\left[x-\frac{2}{3}y\right]^3 = (x)^3 - \left(\frac{2}{3}y\right)^3 - 3(x)\left(\frac{2}{3}y\right)\left(x-\frac{2}{3}y\right)$

$= x^3 - \frac{8}{27}y^3 - 2xy\left(x-\frac{2}{3}y\right)$

$= x^3 - \frac{8}{27}y^3 - 2x^2y + \frac{4}{3}xy^2$

7. Evaluate the following using suitable identities:

$(i)(99)^3$

$Sol: (x - y)^3 \equiv x^3 - y^3 - 3xy\,(x - y)$

$(99)^3 = (100-1)^3$

$= 100^3 - 1^3 - 3(100)(1)[100-1]$

$= 1000000 - 1 - 300(99)$

$= 1000000 - 1 - 29700$

$= 9{,}70{,}299$

$\boldsymbol{(ii)(102)^3}$

$Sol: (x + y)^3 \equiv x^3 + y^3 + 3xy\,(x + y)$

$(102)^3 = (100 + 2)^3$

$= (100)^3 + (2)^3 + 3(100)(2)[100 + 2]$

$= 1000000 + 8 + 600(102)$

$= 1000000 + 8 + 61200$

$= 10{,}61{,}208$

$\boldsymbol{(iii)(998)^3}$

$Sol: (x - y)^3 \equiv x^3 - y^3 - 3xy\,(x - y)$

$(998)^3 = (1000 - 2)^3$

$= (1000)^3 - (2)^3 - 3(1000)(2)[1000 - 2]$

$= 1000000000 - 8 - 6000(998)$

$= 1000000000 - 8 - 5988000$

$= 99{,}40{,}11{,}992$

8. **Factorise each of the following**:

$\boldsymbol{(i)\ 8a^3 + b^3 + 12a^2b + 6ab^2}$

$Sol: :\ x^3 + 3x^2y + 3xy^2 + y^3 = (x + y)^3$

$8a^3 + b^3 + 12a^2b + 6ab^2$

$= (2a)^3 + (b)^3 + 3(2a)^2b + 3(2a)(b)^2$

$= (2a + b)^3$

$\boldsymbol{(ii) 8a^3 - b^3 - 12a^2b + 6ab^2}$

$Sol: x^3 - y^3 - 3x^2y + 3xy^2 \equiv (x - y)^3$

$8a^3 - b^3 - 12a^2b + 6ab^2$

$= (2a)^3 - (b)^3 - 3(2a)^2b + 3(2a)(b)^2$

$= (2a - b)^3$

$(iii)\ 27 - 125a^3 - 135a + 225a^2$

$Sol: \ x^3 - y^3 - 3x^2y + 3xy^2 = (x - y)^3$

$27 - 125a^3 - 135a + 225a^2$

$= (3)^3 - (5a)^3 - 3 \times 3^2 \times 5a + 3 \times 3 \times (5a)^2$

$= (3 - 5a)^3$

$(vi) \quad 64a^3 - 27b^3 - 144a^2b + 108ab^2$

$Sol: \ x^3 - y^3 - 3x^2y + 3xy^2 = (x - y)^3$

$64a^3 - 27b^3 - 144a^2b + 108ab^2$

$= (4a)^3 - (3b)^3 - 3(4a)^2(3b) + 3(4a)(3b)^2$

$= (4a - 3b)^3$

$(v) 27p^3 - \frac{1}{216} - \frac{9}{2}p^2 + \frac{1}{4}p$

$Sol: x^3 - y^3 - 3x^2y + 3xy^2 = (x - y)^3$

$$27p^3 - \frac{1}{216} - \frac{9}{2}p^2 + \frac{1}{4}p = (3p)^3 - \left(\frac{1}{6}\right)^3 - 3(3p)^2\left(\frac{1}{6}\right) + 3(3p)\left(\frac{1}{6}\right)^2 = \left(3p - \frac{1}{6}\right)^3$$

9. Verify (i) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ (ii) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Sol: (i) $\text{R.H.S} = (x + y)(x^2 - xy + y^2) = x(x^2 - xy + y^2) + y(x^2 - xy + y^2)$

$= x^3 - x^2y + xy^2 + x^2y - xy^2 + y^3 = x^3 + y^3 = \text{L.H.S}$

(i) $\text{R.H.S} = (x - y)(x^2 + xy + y^2) = x(x^2 + xy + y^2) - y(x^2 + xy + y^2)$

$= x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3 = x^3 - y^3 = \text{L.H.S}$

10. (i)Factorise $27y^3 + 125z^3$

Sol: (i) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$27y^3 + 125z^3 = (3y)^3 + (5z)^3 = (3y + 5z)[(3y)^2 - (3y)(5z) + (5z)^2]$

$= (3y + 5z)(9y^2 - 15yz + 25z^2)$

(ii) *Factorise* : $64m^3 - 343n^3$

$Sol: x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

$64m^3 - 343n^3 = (4m)^3 - (7n)^3 = (4m - 7n)[(4m)^2 + (4m)(7n) + (7n)^2]$

$= (4m - 7n)(16m^2 + 28mn + 49n^2)$

11. Factorise : $27x^{33} + y^3 + z^3 - 9xyz$

Sol: $x^3 + y^3 + z^3 - 3xyz \equiv (x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz)$

$27x^3 + y^3 + z^3 - 9xyz = (3x)^3 + y^3 + z^3 - 3(3x)(y)(z)$

$= (3x + y + z)[(3x)^2 + y^2 + z^2 - (3x)(y) - (y)(z) - (z)(3x)]$

$= (3x + y + z)(9x^2 + y^2 + z^2 - 3xy - yz - 3xz)$

12. Verify that $x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2]$

Sol: $x^3 + y^3 + z^3 - 3xyz$

$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz)$

$= (x + y + z)\frac{1}{2}(2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2xz)$

$= (x + y + z)\frac{1}{2}[(x^2 - 2xy + y^2) + (y^2 - 2yz + z^2) + (z^2 - 2xz + x^2)]$

$= (x + y + z)\frac{1}{2}[(x - y)^2 + (y - z)^2 + (z - x)^2]$

13. If $x + y + z = 0$, show that $x^3 + y^3 + z^3 = 3xyz$.

Sol: We know that

$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz)$

If $x + y + z = 0$ then

$x^3 + y^3 + z^3 - 3xyz = (0)(x^2 + y^2 + z^2 - xy - yz - xz)$

$x^3 + y^3 + z^3 - 3xyz = 0$

$x^3 + y^3 + z^3 = 3xyz$

14. Without actually calculating the cubes, find the value of each of the following:

(i) $(-\mathbf{12})^3 + (\mathbf{7})^3 + (\mathbf{5})^3$

Sol: Let $x = -12, y = 7, z = 5$

$x + y + z = -12 + 7 + 5 = 0$

We know that if $x + y + z = 0$ then $x^3 + y^3 + z^3 = 3xyz$

$\Rightarrow (-12)^3 + (7)^3 + (5)^3 = 3(-12)(7)(5)$

$\Rightarrow (-12)^3 + (7)^3 + (5)^3 = -1260$

$(\boldsymbol{ii})$ $(\mathbf{28})^3 + (-\mathbf{15})^3 + (-\mathbf{13})^3$

$Sol: Let\ x = 28, y = -15, z = -13$

$x + y + z = 28 - 15 - 13 = 0$

We know that if $x + y + z = 0$ then $x^3 + y^3 + z^3 = 3xyz$

$\Rightarrow (28)^3 + (-15)^3 + (-13)^3 = 3(28)(-15)(-13)$

$\Rightarrow (28)^3 + (-15)^3 + (-13)^3 = 16380$

15. Give possible expressions for the length and breadth of each of the following rectangles, in which their areas are given:

(i)Area: $25a^2 - 35a + 12$

Sol: $25a^2 - 35a + 12 = 25a^2 - 15a - 20a + 12$

$= 5a(5a - 3) - 4(5a - 3)$

$= (5a - 3)(5a - 4)$

Length$=5a - 3$ and Breadth$= 5a - 4$

(ii)Area: $35y^2 + 13y - 12$

Sol: $35y^2 + 13y - 12 = 35y^2 + 28y - 15y - 12$

$= 7y(5y + 4) - 3(5y + 4)$

$= (5y + 4)(7y - 3)$

Length$=5y + 4$ and Breadth$= 7y - 3$

16. What are the possible expressions for the dimensions of the cuboids whose volumes are given below?

(i)Volume : $3x^2 - 12x$

Sol: $3x^2 - 12x = 3x(x - 4)$

$length = 3, \quad breadth = x, \quad height = x - 4$

(ii) $12ky^2 + 8ky - 20k$

Sol: $12ky^2 + 8ky - 20k =$

$= 4k(3y^2 + 2y - 5)$

$= 4k(3y^2 - 3y + 5y - 5)$

$= 4k[3y(y - 1) + 5(y - 1)]$

$= 4k(3y + 5)(y - 1) = l \times b \times h$

$length = 4k, \quad breadth = (3y + 5), \quad height = (y - 1)$