APPENDIX 1

IX-MATHEMATICS-NCERT-2023-24

PROOFS IN MATHEMATICS (notes)

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1. In mathematics, a statement is only acceptable if it is either always true or always false.
2. The first known proof is believed to have been given by the Greek philosopher and mathematician Thales.
3. A 'statement' is a sentence which is not an order or an exclamatory sentence. And, of course, a statement is not a question.
4. A mathematical statement cannot be ambiguous.
5. In mathematics, a statement is only acceptable or valid, if it is either true or false.
6. The example given to show a statement is false is called a counter-example.

Example 1 : State whether the following statements are always true, always false or ambiguous. Justify your answers.

(i) There are 8 days in a week.

Sol: This statement is always false, since there are 7 days in a week,

(ii) It is raining here.

Sol: This statement is ambiguous, since it is not clear where 'here' is.

(iii) The sun sets in the west.

Sol: This statement is always true. The sun sets in the west no matter where we live.

(iv) Gauri is a kind girl.

Sol: This statement is ambiguous, since it is subjective–Gauri may be kind to some and not to others.

(v) The product of two odd integers is even.

Sol: This statement is always false. The product of two odd integers is always odd.

(vi) The product of two even natural numbers is even.

Sol: This statement is always true. However, to justify that it is true we need to do some work.

Example 2 : State whether the following statements are true or false:

(i) The sum of the interior angles of a triangle is 180°.

Sol: This statement is true.

(ii) Every odd number greater than 1 is prime.

Sol: This statement is false.

For example, 9 is an odd number greater than 1 is not a prime number.

(iii) For any real number $x, 4x + x = 5x$.

Sol: This statement is true.

(iv) For every real number $x, 2x > x$.

Sol: This statement is false.

For example, $2 \times (-1) = -2$, and -2 is not greater than -1

(v) For every real number $x, x^2 \geq x$.

Sol: This statement is false.

For example $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, and $\frac{1}{4} \ngtr \frac{1}{2}$

(vi) If a quadrilateral has all its sides equal, then it is a square.

Sol: This statement is false, since a rhombus has equal sides but need not be a square.

Example 3 : Restate the following statements with appropriate conditions, so that they become true statements.

(i) For every real number $x, 2x > x$.

Sol: $If\ x > 0, then\ 2x > x$

(ii) For every real number $x, x^2 \geq x$.

Sol: $If\ x \leq 0\ or\ x \geq 1, then\ x^2 \geq x$.

(iii) If you divide a number by itself, you will always get 1.

Sol: If you divide a number except zero by itself, you will always get 1.

(iv) The angle subtended by a chord of a circle at a point on the circle is 90°.

Sol: The angle subtended by a diameter of a circle at a point on the circle is 90°

(v) If a quadrilateral has all its sides equal, then it is a square.

Sol: If a quadrilateral has all its sides and interior angles equal, then it is a square.

EXERCISE A1.1

1. State whether the following statements are always true, always false or ambiguous. Justify your answers.

(i) There are 13 months in a year.

Sol: False. There are 12 months in a year.

(ii) Diwali falls on a Friday.

Sol: Ambiguous. In a given year, Diwali may or may not fall on a Friday.

(iii) The temperature in Magadi is 26^0 C.

Sol: Ambiguous. At some time in the year, the temperature in Magadi, may be 26° C.

(iv) The earth has one moon.

Sol: Always true

(v) Dogs can fly.

Sol: False. Dogs cannot fly.

(vi) February has only 28 days.

Sol: Ambiguous. In a leap year, February has 29 days.

2. State whether the following statements are true or false. Give reasons for your answers.

(i) The sum of the interior angles of a quadrilateral is 350°.

Sol: False. The sum of the interior angles of a quadrilateral is 360°.

(ii) **For any real number $x, x^2 \geq 0$.**

Sol: True

(iii) A rhombus is a parallelogram.

Sol: True

(iv) The sum of two even numbers is even.

Sol: True.

(v) The sum of two odd numbers is odd.

Sol: False, for example, $3 + 5 = 8$, which is not an odd number

3. Restate the following statements with appropriate conditions, so that they become true statements.

(i) All prime numbers are odd.

Sol: All prime numbers greater than 2 are odd.

(ii) Two times a real number is always even.

Sol: Two times a natural number is always even.

(iii) For any $x, 3x + 1 > 4$.

Sol: For any $x > 1$, $3x + 1 > 4$.

(iv) For any x, $x^3 \geq 0$.

Sol: For any $x \geq 0, x^3 \geq 0$.

(v) In every triangle, a median is also an angle bisector.

Sol: In an equilateral triangle, a median is also an angle bisector.

Deductive Reasoning

The main logical tool used in establishing the truth of an unambiguous statement is deductive reasoning.

EXERCISE A1.2

1. Use deductive reasoning to answer the following:

(i) Humans are mammals. All mammals are vertebrates. Based on these two statements, what can you conclude about humans?

Sol: Humans are vertebrates.

(ii) Anthony is a barber. Dinesh had his hair cut. Can you conclude that Antony cut Dinesh's hair?

Sol: No, Dinesh could have got his hair cut by anybody else.

(iii) Martians have red tongues. Gulag is a Martian. Based on these two statements, what can you conclude about Gulag?

Sol: Gulag has a red tongue.

(iv) If it rains for more than four hours on a particular day, the gutters will have to be cleaned the next day. It has rained for 6 hours today. What can we conclude about the condition of the gutters tomorrow?

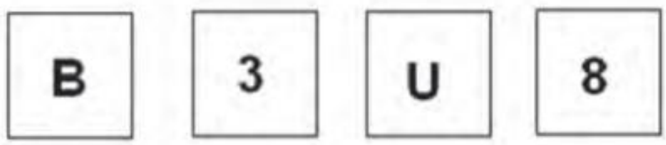
Sol: We conclude that the gutters will have to be cleaned tomorrow.

(v) What is the fallacy in the cow's reasoning in the cartoon below?

Sol: All animals having tails need not be dogs. For example, animals such as buffaloes, monkeys, cats, etc. have tails but are not dogs.

2. Once again you are given four cards. Each card has a number printed on one side and a letter on the other side. Which are the only two cards you need to turn over to check whether the following rule holds?

"If a card has a consonant on one side, then it has an odd number on the other side."



Sol: You need to turn over B and 8. If B has an even number on the other side, then the rule has been broken. Similarly, if 8 has a consonant on the other side, then the rule has been broken.

A1.4 Theorems, Conjectures and Axioms

A mathematical statement whose truth has been established (proved) is called a theorem.

Theorem A1.1 : The sum of the interior angles of a triangle is 180º.

Theorem A1.2 : The product of two even natural numbers is even.

Theorem A1.3 : The product of any three consecutive even natural numbers is divisible by 16.

A conjecture is a statement which we believe is true, based on our mathematical understanding and experience, that is, our mathematical intuition.

Example 4 : Take any three consecutive even numbers and add them, say, $2 + 4 + 6 = 12$, $4 + 6 + 8 = 18$, $6 + 8 + 10 = 24$, $8 + 10 + 12 = 30$, $20 + 22 + 24 = 66$. Is there any pattern you can guess in these sums? What can you conjecture about them?

Sol: One conjecture could be :

(i) The sum of three consecutive even numbers is even.

Another could be :

(ii) The sum of three consecutive even numbers is divisible by 6.

Example 5 : Consider the following pattern of numbers called the Pascal's Triangle:

Line												Sum of numbers
1						1						1
2					1		1					2
3				1		2		1				4
4			1		3		3		1			8
5		1		4		6		4		1		16
6	1		5		10		10		5		1	32
7			:						:			:
8			:						:			:

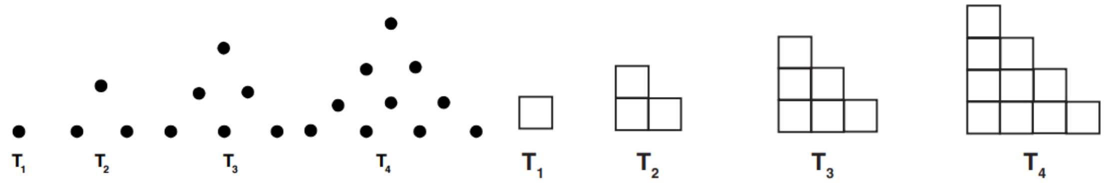
What can you conjecture about the sum of the numbers in Lines 7 and 8? What about the sum of the numbers in Line 21? Do you see a pattern? Make a guess about a formula for the sum of the numbers in line n.

Sol : Sum of the numbers in Line 7 $= 2 \times 32 = 64 = 2^6$

Sum of the numbers in Line 8 $= 2 \times 64 = 128 = 2^7$

Sum of the numbers in Line 21 = 2^{20} Sum of the numbers in Line n = 2^{n-1}

Example 6 : Consider the so-called triangular numbers T_n :



The dots here are arranged in such a way that they form a triangle. Here $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$, and so on. Can you guess what T_5 is? What about T_6 ? What about T_n ? Make a conjecture about Tn.

Sol: $T_5 = 1 + 2 + 3 + 4 + 5 = 15 = \frac{5 \times 6}{2}$

$T_6 = 1 + 2 + 3 + 4 + 5 + 6 = 21 = \frac{6 \times 7}{2}$

$T_n = \frac{n \times (n+1)}{2}$

Conjecture: The statement has not been proved to be true or false.

Goldbach conjecture: "Every even integer greater than 4 can be expressed as the sum of two odd primes."

Theorem: A theorem is a mathematical statement whose truth has been logically established.

EXERCISE A1.3

1. **Take any three consecutive even numbers and find their product; for example, $2 \times 4 \times 6 = 48$, $4 \times 6 \times 8 = 192$, and so on. Make three conjectures about these products.**

Sol: Three possible conjectures are:

(i) The product of any three consecutive even numbers is even.

(ii) The product of any three consecutive even numbers is divisible by 4.

(iii) The product of any three consecutive even numbers is divisible by 6.

2. **Go back to Pascal's triangle.**

Line 1 : 1 = 11^0

Line 2 : 1 1 = 11^1

Line 3 : 1 2 1 = 11^2

Make a conjecture about Line 4 and Line 5. Does your conjecture hold? Does your conjecture hold for Line 6 too?

Sol: Line 4: 1 3 3 1 = 11^3 ;

Line 5: 1 4 6 4 1 = 11^4;

The conjecture holds for Line 4 and Line 5;

No, the conjecture does not holds for Line 6 because $11^5 \neq 15101051$. $(11^5 = 161051)$

3. **Let us look at the triangular numbers (see Fig.A1.2) again. Add two consecutive triangular numbers. For example, $T_1 + T_2 = 4$, $T_2 + T_3 = 9$, $T_3 + T_4 = 16$.**

   What about $T_4 + T_5$? Make a conjecture about $T_{n-1} + T_n$.

Sol: $T_4 + T_5 = 25 = 5^2$

$T_{n-1} + T_n = n^2$

4. **Look at the following pattern:**

   $\mathbf{1^2 = 1}$

   $\mathbf{11^2 = 121}$

   $\mathbf{111^2 = 12321}$

   $\mathbf{1111^2 = 1234321}$

   $\mathbf{11111^2 = 123454321}$

   Make a conjecture about each of the following:

   $\mathbf{111111^2 =}$

   $\mathbf{1111111^2 =}$

   Check if your conjecture is true.

Sol: Conjecture: " $(1111 \ldots . n - \text{times})^2 = 123 \ldots .(n-1)n(n-1) \ldots .321$

$111111^2 = 12345654321$

$1111111^2 = 1234567654321$

The conjecture is true.

5. **List five axioms (postulates) used in this book.**

Sol: Euclid's axioms

(1) Things which are equal to the same thing are equal to one another.

(2) If equals are added to equals, the wholes are equal.

(3) If equals are subtracted from equals, the remainders are equal.

(4) Things which coincide with one another are equal to one another.

(5) The whole is greater than the part.

(6) Things which are double of the same things are equal to one another.

(7) Things which are halves of the same things are equal to one another.

Euclid's five postulates

(i) A straight line may be drawn from any one point to any other point

(ii) A terminated line can be produced indefinitely.

(iii) A circle can be drawn with any centre and any radius.

(iv) All right angles are equal to one another.

(v) If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles.

A1.5 :What is a Mathematical Proof?

A process which can establish the truth of a mathematical statement based purely on logical arguments is called a mathematical proof.

To show that a mathematical statement is false, it is enough to find a single counter-example.

Ex: $7 + 5 = 12$ is a counter-example to the statement that the sum of two odd numbers is odd.

The list of basic ingredients in a proof:

(i) To prove a theorem, we should have a rough idea as to how to proceed.

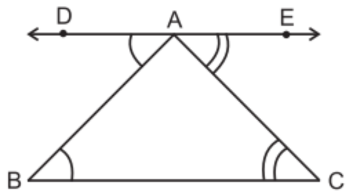
(ii) The information already given to us in a theorem (i.e., the hypothesis) has to be clearly understood and used.

(iii) A proof is made up of a successive sequence of mathematical statements.

Each statement in a proof is logically deduced from a previous statement in the proof, or from a theorem proved earlier, or an axiom, or our hypothesis.

(iv) The conclusion of a sequence of mathematically true statements laid out in a logically correct order should be what we wanted to prove, that is, what the theorem claims.

Theorem A1.1 : The sum of the interior angles of a triangle is 180°.



Proof : Consider a triangle ABC.

RTP: $\angle ABC + \angle BCA + \angle CAB = 180°$

Construction: Draw a line DE parallel to BC passing through A.

Proof: DE is parallel to BC and AB is a transversal.

$\angle DAB = \angle ABC$ (alternate interior angles are equal)$\rightarrow$(1)

Similarly, $\angle CAE = \angle ACB \rightarrow$(2)

$\angle$ BAC= $\angle$ BAC $\rightarrow$(3)

From (1)+(2)+(3)

$\angle$ ABC + $\angle$ BAC+ $\angle$ ACB = $\angle$ DAB +$\angle$ BAC + $\angle$ CAE (From Euclid's axiom)

But $\angle$ DAB +$\angle$ BAC + $\angle$ CAE = 180° , since they form a straight angle.

Hence, $\angle$ ABC + $\angle$ BAC+ $\angle$ ACB = 180°.

Hence proved.

Theorem A1.2 : The product of two even natural numbers is even.

Proof : Let x and y be any two even natural numbers.

We want to prove that xy is even.

$x = 2m$ and $y = 2n$, for some natural numbers m and n.

$xy = 2m \times 2n = 4mn = 2 \times 2mn =$ even number

$xy \text{ is even}.$

Theorem A1.3 : The product of any three consecutive even natural numbers is divisible by 16.

Proof: Let the three consecutive even numbers are 2n, 2n + 2 and 2n + 4, for some natural number n.

$$2n(2n + 2)(2n + 4) = 2n \times 2(n + 1) \times 2(n + 2)$$

$$= 2 \times 2 \times 2n(n + 1)(n + 2)$$

$$= 8n(n + 1)(n + 2).$$

Case1: If n is even then we can write $n = 2m$, for some natural number m

$$8n(n + 1)(n + 2) = 8 \times 2m \times (2m + 1)(2m + 2)$$

$$= 16m(2m + 1)(2m + 2) \text{ is divisible by } 16$$

Case1: If n is odd then we can write $n = 2m + 1$, for some natural number m

$$8n(n + 1)(n + 2) = 8 \times (2m + 1) \times (2m + 1 + 1)(2m + 1 + 2)$$

$$= 8 \times (2m + 1) \times (2m + 2)(2m + 3)$$

$= 8 \times (2m + 1) \times 2 \times (m + 1)(2m + 3)$

$= 16(2m + 1)(m + 1)(2m + 3)$ is divisible by 16

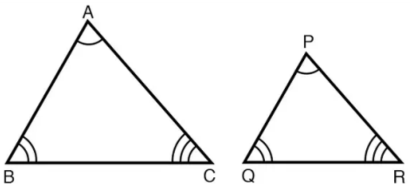
So, in both cases we have shown that the product of any three consecutive even numbers is divisible by 16.

EXERCISE A1.4

1. Find counter-examples to disprove the following statements:

(i) If the corresponding angles in two triangles are equal, then the triangles are congruent.

Sol:



The corresponding angles in two triangles are equal but of different sides.

(ii) A quadrilateral with all sides equal is a square.

Sol: A rhombus has equal sides but may not be a square.

(iii) A quadrilateral with all angles equal is a square.

Sol: A rectangle has equal angles but may not be a square.

(iv) **For integers a and b, $\sqrt{a^2 + b^2} = a + b$**

Sol: For a = 3 and b = 4

$\sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5; a + b = 3 + 4 = 7$

Counter example is a = 3 and b = 4

(v) $2n^2 + 11$ is a prime for all whole numbers n.

Sol: Let $n = 11$

$2n^2 + 11 = 2(11)^2 + 11 = 2 \times 121 + 11 = 242 + 11$

$= 253$ ($= 11 \times 23$)is not a prime number.

Counter example is $n = 11$

(vi) **$n^2 - n + 41$ is a prime for all positive integers n.**

Sol: For n=41

$n^2 - n + 41 = (41)^2 - 41 + 41$

$= (41)^2 = 41 \times 41$ *is not a prime number*.

Counter example is $n = 41$

2. Take your favourite proof and analyse it step-by-step along the lines discussed in Section A1.5 (what is given, what has been proved, what theorems and axioms have been used, and so on).

3. Prove that the sum of two odd numbers is even.

Sol: Let x and y be two odd numbers.

Then $x = 2m + 1$ and $y = 2n + 1$ for some natural numbers m and n

Sum $= x + y = 2m + 1 + 2n + 1 = 2m + 2n + 2 = 2(m + n + 1)$ is divisible by 2.

$x + y$ is even.

4. **Prove that the product of two odd numbers is odd.**

Sol: Let x and y be two odd numbers.

Then $x = 2m + 1$ and $y = 2n + 1$ for some natural numbers m and n

Product $= xy = (2m + 1)(2n + 1) = 2mn + 2m + 2n + 1$
$= 2(mn + m + n) + 1$ is not divisible by 2.

xy is odd

5. **Prove that the sum of three consecutive even numbers is divisible by 6.**

Sol: Let the three consecutive even numbers are 2n, 2n + 2 and 2n + 4, for some natural number n.

$Sum = 2n + 2n + 2 + 2n + 4 = 6n + 6 = 6(n + 1)\ is\ divisible\ by\ 6.$

$\therefore$ The sum of three consecutive even numbers is divisible by 6.

6. **Prove that infinitely many points lie on the line whose equation is y = 2x.**

Sol: Any point on the given line y=2x is (n,2n) for many natural number n.

The points are (1,2), (2,4), (-1,-2),……..

7. **You must have had a friend who must have told you to think of a number and do various things to it, and then without knowing your original number, telling you what number you ended up with. Here are two examples. Examine why they work.**

(i) Choose a number. Double it. Add nine. Add your original number. Divide by three. Add four. Subtract your original number. Your result is seven.

Sol: Let your original number be n.

$n \to 2n \to 2n + 9 \to 2n + 9 + n = 3n + 9 \to \frac{3(n+3)}{3} = n + 3$

$\to n + 3 + 4 = n + 7 \to n + 7 - n = 7$

(ii) Write down any three-digit number (for example, 425). Make a six-digit number by repeating these digits in the same order (425425). Your new number is divisible by 7, 11 and 13

Sol: 425425=425×1001=425×$(7 \times 11 \times 13)$

425425 is divisible by 7, 11 and 13

Take any three digit number say, abc

$abcabc = abc \times 1001 = abc \times (7 \times 11 \times 13)$

$abcabc$ is divisible by $7, 11$ and 13.